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On Multiple Instanton-antiinstanton Configurations

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We show why and how the $I\bar{I}$ valley trajectory commonly used in the literature so far is in fact unsatisfactory. A better $I\bar{I}$ valley is suggested. We also give analytic expressions for the multiple instanton-antiinstanton configurations in the pure Yang-Mills theory. These formulas make it possible to go beyond the dilute gas approximation and calculate the multi-body interactions among instantons and antiinstantons.

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1. Introduction

In the Euclidean version of the 4-dimensional pure Yang-Mills theory, with the action

$$S = \frac{1}{2g^2} \int d^4x \, tr F^2, \tag{1.1}$$

field configurations which correspond to finite values of the action fall into discrete sectors characterized by an integer Q, the Pontryagin index. In each sector, the solution to the field equation is exactly known and shown to be unique. They are the N-instanton (I^N) solutions[1–3][1], with $N = Q$. The I^N solution is parametrized by $(8N - 3)$ independent degrees of freedom, which we shall denote as ω. One can interpret them as the positions (4 for each instanton), the sizes (1 each) and the group orientations (or phases, 3 each, minus the 3 overall phases which can be undone by the global gauge transformations). If we are interested in the Q-sector contribution to the path integral,

$$Z_Q = \int_{A \in Q \text{sector}} [\mathcal{D}A] \exp(-S[A]), \tag{1.2}$$

the I^N will dominate because it minimizes the action. Furthermore, those $(8n - 3)$ zero modes should be isolated from the other degrees of freedom using the collective coordinate method, together with the 3 global gauge transformations. Keeping only the supposedly dominant exponent[2], we have

$$Z_Q \sim \frac{1}{Q!} \int d^{8Q}\omega \ \exp(-S(\omega)). \tag{1.3}$$

The original field theory problem is thus reduced to that of interacting particles. In this specific case, the action S is well known, *i.e.*

$$S_{(I^N)} = NS_I = \frac{8\pi^2}{g^2}N, \tag{1.4}$$

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[1] We will adopt the quaternion notation used in ref.1. For a brief introduction, see Appendix A.

[2] This is a very crude approximation. It gives only part of the leading contribution in the semi-classical expansion. This is adequate for our purpose, however. A general treatment for the complete leading term of the semi-classical approximation can be found in our other paper[4]. The explicit calculation for the one-instanton case was first carried out by 't Hooft[5] and can be found in numerous reviews.

independent of ω. In other words, it consists purely of instanton "self-action", and there is no interaction among instantons. As for the path integral, it now becomes

$$Z_Q \sim \frac{1}{Q!} \left\{ \int d^8\omega \exp(-S_I) \right\}^Q. \tag{1.5}$$

We have ignored those sub-dominant configurations which are not solutions to the field equation. Their contributions may be important sometimes and should be included in our approximation. The most important of these sub-dominant configurations is the N-instanton-$\bar{N}$-antiinstanton $(I^N \bar{I}^{\bar{N}})$, with $N - \bar{N} = Q$. For widely separated $I^N \bar{I}^{\bar{N}}$, the interactions are negligible and we again have

$$S_{(I^N \bar{I}^{\bar{N}})} \sim (N + \bar{N})S_I. \tag{1.6}$$

Therefore,

$$Z_Q \sim \sum_{N,\bar{N}} \frac{\delta_{N-\bar{N}-Q}}{N!\bar{N}!} \int d^{8(N+\bar{N})}\omega \ \exp(-S(\omega)). \tag{1.7}$$

Using the identity

$$\delta_n = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{-in\theta}, \tag{1.8}$$

we can further simplify (1.7),

$$\begin{aligned} Z_Q &\sim \int_0^{2\pi} d\theta \frac{e^{iQ\theta}}{2\pi} \sum_{N,\bar{N}} \frac{1}{N!\bar{N}!} \int d^{8(N+\bar{N})}\omega \ \exp(-N(S_I + i\theta) - \bar{N}(S_I - i\theta)) \\ &\sim \int_0^{2\pi} d\theta \frac{e^{iQ\theta}}{2\pi} \exp\left(\int d^8\omega \ e^{-S_I} \ \cos\theta \right). \end{aligned} \tag{1.9}$$

We have presented a simplified version of the so-called dilute-instanton-gas calculation. There are three possible improvements over (1.9). Firstly, one can incorporate the pre-exponential factor so that the result becomes the true leading term in the semi-classical expansion. This has been done for $Q = 1$ in ref.5. Secondly, one may want to improve (1.6) by introducing instanton-antiinstanton interactions. This will be the main goal of this paper. Lastly, since the integral $\int d^8\omega$ contains the integration over the instanton size ρ, the semi-classical (small g) approximation naturally breaks down at the infrared limit, as a result of the renormalization group running effect. This is a common problem that plagues all semi-classical treatments for the 4-dimensional pure Yang-Mills theory. We are unable to provide new insights into this problem. However, this difficulty is a totally

separate issue from the dilute-gas approximation we will try to improve, and should not invalidate our treatments. If the theory contains a scale cutoff as a result of the Higgs mechanism or finite temperatures, the infrared problem is circumvented and our multiple instanton-antiinstanton results will be valid. For simplicity, we will avoid for now the complexity involved in the Yang-Mills-Higgs system, which will be discussed in our future paper[4], and concentrate on the pure Yang-Mills theory instead.

There have been some previous efforts trying to go beyond the dilute-instanton-gas approximation. Callan, Dashen and Gross[6] were the first to compute the leading $I\bar{I}$ interaction at the large separation (R) limit. Their result, however, is not conformally invariant. It is also very difficult to calculate subleading terms using their method. Superior in both aspects is the later work by Yung[7]. Using a spherical ansatz, he reduced the Yang-Mills action to that of a quantum mechanical double-well. This trick enabled him to write down the $I\bar{I}$ configuration in the Yang-Mills theory corresponding to the kink-antikink in the double-well system, and the $I\bar{I}$ interaction to all orders in ρ/R simply followed. We will review this important result in detail in Section 2.

Elegant though Yung's solution is, it relies heavily on the coincidence which connects the Yang-Mills theory with the simpler quantum mechanical system, which in turn relies on the spherical ansatz. Therefore this method obviously cannot be generalized to anything more complex than $I\bar{I}$. Employing a brand new philosophy, we construct a systematic treatment that will make it possible to find expressions for $I^N \bar{I}^{\bar{N}}$. In Section 3, we illustrate this method in the simplest case of $I\bar{I}$. Surprisingly, Yung's solution will be shown to be unsatisfactory. This is an important result because naive application of Yung's valley formula has been heavily used to compute the high-energy baryon-number violating cross-section in the standard model. Improvement on the understanding of the valley trajectory can dispel some common misconceptions. In Section 4, we generalize our result to $I^2 \bar{I}^2$ and beyond. Although these semi-classical results do not have direct applications in QCD at this moment (except maybe for the instanton-liquid hypothesis), they are nonetheless interesting not only because they provide corrections to the dilute-instanton-gas approximation, but also because they can serve as a primer for similar treatments for the Yang-Mills-Higgs system. It has been argued by some authors[8,9] that, again, in the high-energy instanton-induced baryon-number violating processes, the multi-instanton effects become important long before the one-instanton amplitude has grown large. Their analysis relied, however, on a crude nearest neighbor approximation, and was questioned by other authors[10,11]. This controversy clearly cannot be settled until we gain better knowledge of the multi-instanton configurations in the Yang-Mills-Higgs theory, and our result should be the first step toward achieving this goal.

2. Yung's Valley Solution for $I\bar{I}$

Before we introduce Yung's result, it is necessary to familiarize ourselves with the conformal properties of the instantons. This is because the 4-dimensional Yang-Mills lagrangian is classically invariant under the conformal group, which includes the Poincare group as well as the dilatation and four special conformal transformations. Together with the global gauge transformations, they ensure that all 8 parameters of the one-instanton solution (I) correspond to zero modes. One can apply this group theory analysis to the two-instanton solution[3] (I^2) also, and find that, of the 16 parameters, all but two have to be zero modes due to these symmetries. Although one can show that even these two potential exceptions turn out to be zero modes, either by direct computation or by using a much more involved argument than we care to reproduce here, it is still interesting to find these two modes explicitly. After some tedious calculation, we find that they correspond to the relative phase and a dimensionless parameter $z_2 = (R^2 + \rho_1^2 + \rho_2^2)^3/(R^2\rho_1^2\rho_2^2)$ which can be interpreted as the separation between the two instantons.

Since the $I\bar{I}$ configuration should also be described similarly by 16 parameters, it is natural to wonder what insight we can get using the group theory argument. This turns out to be more difficult than one would imagine because of the uncertainty involved in reducing a field configuration with an infinite number of degrees of freedom, to an unknown expression parameterized by only 16. We therefore make the assumption that I and $\bar{I}$ can be put together in a more or less linear manner[4], and find again that all but two correspond to zero modes. These two possible non-zero modes are the relative phase and the parameter,

$$z \;=\; (R^2 + \rho_1^2 + \rho_2^2)/(2\rho_1\rho_2). \tag{2.1}$$

They are invariant under all the conformal transformations. Let us ignore the relative phase for now, and concentrate on the z direction. As z increases from its minimal value 1 to infinity, we produce an instanton-antiinstanton pair from the trivial vacuum and pull them farther and farther apart. Therefore, the action should increase from 0 to $2S_I$ accordingly. This is exactly what makes the instanton-antiinstanton pair important. The

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[3] The standard introductory text for the the I^N solution is by Atiyah[1]. We discuss some interesting properties of I^2 in one of our earlier papers[12].

[4] We are aware that this sounds awfully vague. We do not consider it worthwhile to present this result in detail though, because its importance has been largely diminished by the results we shall present later in this paper.

action flattens out at large separation, and its effects would have been badly accounted for if we had naively treated this mode like any other quantum perturbation. Yung and other authors call this mode the valley direction because it corresponds to a low-lying valley if one considers the action as a functional in the field configuration space. We will also use the phrase "quasi-zero modes" sometimes, partly because they require the collective coordinate treatment, similar to the real zero modes.

We are now ready to present Yung's result. Making full use of the conformal symmetries, we can tranform any given set of $I\bar{I}$ parameters into one which satisfies

$$R = 0, \qquad \rho_1 \rho_2 = 1, \qquad \rho_1 \leq \rho_2. \tag{2.2}$$

This corresponds to an instanton sitting right on top of an antiinstanton of a possibly different size. Therefore, it is natural to make the following spherical ansatz,

$$A_{\text{Yung}} = \text{Im} \left\{ \frac{x d\bar{x}}{x^2} s(x^2) \right\}, \tag{2.3}$$

since both the instanton and the antiinstanton can be put in this form. More specifically, the instanton has to be put in the regular gauge,

$$A_I^{\text{reg}} = \text{Im} \left\{ \frac{x d\bar{x}}{x^2 + \frac{1}{\rho_2^2}} \right\}, \tag{2.4}$$

and the antiinstanton in the singular gauge,

$$A_{\bar{I}}^{\text{sing}} = \text{Im} \left\{ \frac{\rho_2^2 x d\bar{x}}{x^2 (x^2 + \rho_2^2)} \right\}. \tag{2.5}$$

Our next step is to substitute (2.3) into (1.1). Here a miracle occurs, and we find

$$S = \frac{48\pi^2}{g^2} \int_{-\infty}^{\infty} dt \left\{ \frac{1}{2} \left(\frac{ds}{dt} \right)^2 + \frac{1}{2} \left[\left(s - \frac{1}{2} \right)^2 - \frac{1}{4} \right]^2 \right\}, \tag{2.6}$$

where $t = \ln x^2$. As promised earlier, the integral is exactly the action of a quantum mechanical double-well. The instanton (2.4) gives the kink at $-\xi$,

$$s_I^{\xi}(t) = \frac{1}{2} \left[1 + \tanh \left(\frac{1}{2} (t + \xi) \right) \right], \tag{2.7}$$

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where $\xi = \ln \rho_2^2$, and the antiinstanton (2.5) gives the antikink at ξ,

$$s_{\bar{I}}^{\xi}(t) = \frac{1}{2}\left[1 - \tanh\left(\frac{1}{2}(t - \xi)\right)\right]. \tag{2.8}$$

Such one-to-one correspondences are encouraging, and one is naturally tempted to use the kink-antikink configuration for $I\bar{I}$. We then have

$$\begin{aligned} s &= \frac{1}{2}\left[\tanh\left(\frac{1}{2}(t+\xi)\right) - \tanh\left(\frac{1}{2}(t-\xi)\right)\right] \\ &= \frac{x^2}{x^2 + \rho_1^2} - \frac{x^2}{x^2 + \rho_2^2}. \end{aligned} \tag{2.9}$$

Putting this back into (2.3), we have

$$A_{\text{Yung}}^{r-r} = \text{Im}\left\{\frac{x d\bar{x}}{x^2 + \rho_1^2} - \frac{x d\bar{x}}{x^2 + \rho_2^2}\right\}, \tag{2.10}$$

$$= A_{\text{Yung}}^{s-s} = \text{Im}\left\{-\frac{\rho_1^2 x d\bar{x}}{x^2(x^2 + \rho_1^2)} + \frac{\rho_2^2 x d\bar{x}}{x^2(x^2 + \rho_2^2)}\right\}, \tag{2.11}$$

or, after a gauge transformation,

$$A_{\text{Yung}}^{s+r} = \text{Im}\left\{\frac{\rho_1^2 \bar{x} dx}{x^2(x^2 + \rho_1^2)} + \frac{\bar{x} dx}{x^2 + \rho_2^2}\right\}. \tag{2.12}$$

Notice that since $z = (\rho_1^2 + \rho_2^2)/2$ and $\rho_1 \rho_2 = 1$, A_{Yung} describes the trivial vacuum for $z = 1$ (as can be seen from (2.10)), and an instanton-antiinstanton pair at large separation for $z \to \infty$ (from (2.12)). This is just what one would expect from the $I\bar{I}$ valley. Substituting (2.9) into (2.6), we get the action profile for Yung's $I\bar{I}$ valley,

$$S(A_{\text{Yung}}) = \frac{16\pi^2}{g^2}\left\{\frac{\rho_2^8 - 8\rho_2^4 - 17}{(1 - \rho_2^4)^2} - \frac{36\rho_2^4 + 12}{(1 - \rho_2^4)^3}\ln\rho_2^2\right\}. \tag{2.13}$$

As explained earlier, A_{Yung} is given only for the instanton-antiinstanton pairs satisfying (2.2). The expression for a general instanton-antiinstanton pair with arbitrary $(R^0, \rho_1^0, \rho_2^0)$ is found by conformal-transforming the corresponding A_{Yung} with $z = (\rho_1^2 + \rho_2^2)/2 = (R^{0^2} + \rho_1^{0^2} + \rho_2^{0^2})/(2\rho_1^0\rho_2^0)$. The action for a general instanton-antiinstanton pair is therefore identical to that of the corresponding A_{Yung}, which can be expressed in terms of z by substituting

$$\rho_2^2 = z + \sqrt{z^2 - 1}, \tag{2.14}$$

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into (2.13). We have

$$S_{\text{Yung}}(z) = \frac{16\pi^2}{g^2} \left\{ \frac{2 - 8z^2 + 9z\sqrt{z^2 - 1}}{z^2 - 1} \right.$$
$$\left. + \frac{3\left(2z^3 - (2z^2 + 1)\sqrt{z^2 - 1}\right)}{(z^2 - 1)^{\frac{3}{2}}} \ln\left(z + \sqrt{z^2 - 1}\right) \right\}. \tag{2.15}$$

If this derivation for analytic $I\bar{I}$ expressions seems amazingly simple, it is because we have not mentioned the caveat yet. As is well known, the $I\bar{I}$ valley, or any quasi-zero mode in general, is not a minimum of the action, or equivalently a solution to the field equation,

$$\left. \frac{\delta S(A)}{\delta A} \right|_{A_{I\bar{I}}} = 0. \tag{2.16}$$

Instead, it is the minimum only under constraints which limit the degree of freedom along the valley direction. Therefore, the valley configuration $A_{I\bar{I}}$ is a solution to (2.16) under a certain constraint. Yung considered the following constraint to be natural,

$$\int d^4x \left(A - A_{I\bar{I}}\right) \frac{\partial A_{I\bar{I}}}{\partial z} = 0, \tag{2.17}$$

because the sectors in which the solution $A_{I\bar{I}}$ is a minimum are perpendicular to the valley direction. One therefore has to solve

$$\left. \frac{\delta S(A)}{\delta A} \right|_{A_{I\bar{I}}} \propto \frac{\partial A_{I\bar{I}}}{\partial z}. \tag{2.18}$$

Unfortunately, the Yung form (2.10) or (2.12) does not satisfy (2.18). One is thus forced to consider constraints which cut out sectors not perpendicular to the valley direction, or putting it differently, perpendicular only if one defines a generalized inner product which varies with z. This is why Yung correctly limited the validity of his result to the leading order result in the large z region only. Other authors have been more daring[13]. They claim that with a suitably defined varying inner product, A_{Yung} should be considered a valid valley trajectory for all values of z. This turns out not to be true, as we shall see in the next section.

3. The Valley Method Done Right

Although the correspondence between the Yang-Mills instantons and the kinks in the double-well potential is an amazing fact, it also prevents us from generalizing Yung's method to anything not spherically symmetric. In order to overcome this difficulty, we have to find a way to deal with the Yang-Mills instantons directly. Let us reexamine Yung's derivation for inspiration. Notice that the kink-antikink configuration we used in (2.9) does not satisfy the analog of (2.18) in the double-well system. Instead, it is simply a linear combination of a kink and an antikink. In fact, this is why A_{Yung} does not satisfy (2.18) and requires a redefinition of the inner product. One may wonder if Yang-Mills instantons and antiinstantons can be put together linearly without us bothering with their quantum mechanical counterparts.

Such attempts have been made since the early days of instantons. They inevitably failed because as the instanton-antiinstanton pair gets close to each other, the expression will not gradually approach the trivial vacuum, if one insists on having both in the same gauge, which most of the early authors did. If we reason carefully, however, we find no real reason why this has to be so, other than the fact that it would automatically guarantee the Z_2 spacial reflection symmetry of the lagrangian. We will abandon this reflection symmetry in order to pursue a simple expression for the valley configuration. This expression must satisfy all other good properties one would expect from the instanton-antiinstanton pair. We now list these criteria,

1) $A_{I\bar{I}}$ belongs in the $Q = 0$ sector.
2) $A_{I\bar{I}}$ has easily identifiable instanton parameters, and covers the entire 16-dimensional parameter space spanned by all zero- and nonzero-modes.
3) $A_{I\bar{I}}$ becomes the sum of an instanton and an antiinstanton at large separation, and approaches the trivial vacuum as $z \to 1$.
4) $A_{I\bar{I}}$ respects the symmetries of the theory. This includes the conformal symmetries and a Z_2 symmetry which we will explain in more detail later.

These criteria may seem arbitrary, but in fact they are not. They are all that we know for sure about $A_{I\bar{I}}$. Every other detail in $A_{I\bar{I}}$ can be compensated by the choice of constraints. To see this, recall that $A_{I\bar{I}}$ satisfies (2.16) only after a contraint is applied. If we choose a general linear constraint

$$\int d^4x \, (A - A_{I\bar{I}}) \, f_z(x) \;=\; 0, \tag{3.1}$$

what we need to solve becomes

$$\left.\frac{\delta S(A)}{\delta A}\right|_{A_{I\bar{I}}} \quad \propto \quad f_z(x). \tag{3.2}$$

Instead of fixing the constraint to solve for $A_{I\bar{I}}$, which is always a difficult if not impossible task, we can choose $A_{I\bar{I}}$ first, then use (3.2) to find f_z, which amounts to no more than a simple substitution of $A_{I\bar{I}}$ into the left hand side of (3.2). This is to say that the bottom of the valley is not strictly-defined, and we should make the best use of this freedom.

Before we endeaver to find the expression satisfying all these criteria, let's first examine how A_{Yung}^{s+r} stacks up against them. It satisfies $Cri.1$ and $Cri.2$ quite trivially, although we haven't mentioned how to put in the phases. This is done by sandwiching both the instanton and the antiinstanton with $SU(2)$ group elements, or in our notation, unit quaternion constants a and b, as follows,

$$A_{\text{Yung}}^{s+r} \;=\; \text{Im}\left\{\frac{\rho_1^2 a\bar{x}dx\bar{a}}{x^2(x^2+\rho_1^2)} + \frac{b\bar{x}dx\bar{b}}{x^2+\rho_2^2}\right\}. \tag{3.3}$$

As for $Cri.3$, A_{Yung}^{s+r} satisfies the first part because it is simply a linear combination of the (anti)instantons, and the second part because the (anti)instantons are in the singular and the regular gauge respectively. When $z \to 1$, (2.12) becomes

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\begin{aligned}
A_{\text{Yung}}^{s+r} \;&=\; \text{Im}\left\{\frac{\bar{x}dx}{x^2(x^2+1)} + \frac{\bar{x}dx}{x^2+1}\right\} \\
&=\; \text{Im}\left\{\frac{\bar{x}dx}{x^2}\right\},
\end{aligned} \tag{3.4}
$$

which is a pure-gauge configuration.[5]

So far, A_{Yung}^{s+r} has passed the tests with flying colors. This suggests that it is pretty close to the "true" valley bottom. Unfortunately, as we shall show now, it is not close enough. The problem lies in $Cri.4$. A_{Yung}^{s+r} does respect the conformal symmetries, but this is done in a rather artificial way. Recall that A_{Yung}^{s+r} is defined only under the constraint (2.2). All other configurations are given by conformal projection. Although this seems contrived, it nonetheless gets the job done. It is not so when it comes to the Z_2 symmetry, by which we mean exchanging ρ_1 and ρ_2. Clearly this corresponds to exchanging the

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[5] In fact, this coincidence is more general than this, as we shall see in the next section.

instanton and the antiinstanton, and thus the action should remain unchanged[6]. It turns out that S_{Yung} does not respect this symmetry[7]. The problem is particularly bad for $z \sim 1$. Let's first define

$$\theta = \rho_2 - \rho_1. \tag{3.5}$$

Expanding (2.13) for small θ then gives

$$S_{\text{Yung}} \sim \frac{16\pi^2}{g^2} \left\{ \frac{6}{5}\theta^2 - \frac{4}{5}\theta^3 + \frac{9}{35}\theta^4 + \mathcal{O}\left(\theta^5\right) \right\}. \tag{3.6}$$

The odd power terms clearly violate the Z_2 symmetry. If problems in the third power don't seem too bad, consider the action S_{Yung} for the instanton-antiinstanton pair with opposite phases, i.e. $a\bar{b} + b\bar{a} = 0$. We have

$$S_{\text{Yung}}^{+-} = \frac{16\pi^2}{g^2} \left\{ \frac{\rho_2^4 + 1}{\rho_2^4 - 1} - \frac{4}{(1 - \rho_2^4)^2} \ln \rho_2^2 \right\}. \tag{3.7}$$

This has the small θ expansion,

$$S_{\text{Yung}}^{+-} \sim \frac{16\pi^2}{g^2} \left\{ 2 - \frac{2}{3}\theta + \frac{1}{6}\theta^2 + \mathcal{O}\left(\theta^3\right) \right\}. \tag{3.8}$$

Clearly, A_{Yung}^{s+r} has wandered away from the true valley trajectory a bit too far, especially for small separations.

We now resume our quest for a better expression for the $I\bar{I}$ valley. We still want to use linear combinations of the instanton and the antiinstanton. By now, it should be clear how this can be done. We put one in the singular gauge and the other in the regular gauge. We have

$$A_{I\bar{I}} = \text{Im} \left\{ \frac{\rho_1^2 a\bar{x}dx\bar{a}}{x^2(x^2 + \rho_1^2)} + \frac{b(\bar{x} - R)dx\bar{b}}{(x - R)^2 + \rho_2^2} \right\}. \tag{3.9}$$

Unfortunately, this expression contains some conformal degrees of freedom, and if we substitute it into (1.1), these degrees of freedom do not become zero modes as they should. The brute force solution to this problem is to use a constraint a la Yung to get rid of

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[6] This is a weaker form of the spacial reflection symmetry. Instead of the lagrangian, we only require the action to be invariant.

[7] In fact, it is possible to have S_{Yung} compatible with the Z_2, but this is done by defining A_{Yung} as in Eq.(2.10),(2.11) and (2.12) only for $\rho_1 \leq \rho_2$. One then defines the configurations with $\rho_1 \geq \rho_2$ to be the Z_2 projections of A_{Yung}. Unfortunately, this procedure introduces a discontinuity into S_{Yung} at $z = 1$.

these conformal modes[8]. We will define $A_{I\bar{I}}$ only on the slice cut out by this constraint and then conformally project it to the entire parameter space. For example, if we choose the constraint (2.2), we recover A_{Yung}. There are other obvious choices of constraints, however. For example, we can use

$$\rho_1 \;=\; \rho_2 \;=\; 1. \tag{3.10}$$

This gives

$$A_{I\bar{I}} \;=\; \text{Im}\left\{\frac{a\bar{x}dx\bar{a}}{x^2(x^2+1)} + \frac{b(\overline{x-R})dx\bar{b}}{(x-R)^2+1}\right\}. \tag{3.11}$$

To see if this is compatible with the Z_2 symmetry $\theta \to -\theta$, let's first note that

$$\begin{aligned}
\theta \;&=\; \rho_2 - \rho_1 && \text{under (2.2),} \\
&=\; \sqrt{2(z-1)} && \text{in general,} \\
&=\; R && \text{under (3.10).}
\end{aligned} \tag{3.12}$$

Therefore $\theta \to -\theta$ is equivalent to $R \to -R$, which correponds to moving the $\bar{I}$ from R to $-R$. This can also be achieved by a rotation, which is a perfectly good symmetry of the expression. Thus we expect that (3.11) should respect the Z_2 symmetry in question. Explicit calculation confirms this expectation. For the instanton-antiinstanton pair with opposite phases, *i.e.* $a\bar{b} + b\bar{a} = 0$, the action has the small θ expansion

$$S_{I\bar{I}}^{+-} \;\sim\; \frac{16\pi^2}{g^2}\left\{2 - \frac{1}{3}\theta^2 + \mathcal{O}\left(\theta^4\right)\right\}. \tag{3.13}$$

If the phases are aligned with each other, *i.e.* $a = b$, we have

$$S_{I\bar{I}} \;\sim\; \frac{16\pi^2}{g^2}\left\{\frac{6}{5}\theta^2 - \frac{33}{35}\theta^4 + \mathcal{O}\left(\theta^6\right)\right\}. \tag{3.14}$$

Therefore (3.11) is clearly a better solution than A_{Yung}.

As mentioned before, the valley solution has a dependence on the constraint function $f_z(x)$. It is therefore perfectly plausible for one to discover other equally satisfactory solutions with different constraints. One may ask if there is any reason why he should go through the trouble of looking for such alternative solutions. The answer is yes because

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[8] This constraint gets rid of the zero modes, and leaves only the quasi-zero modes. This should be compared to the constraints defined in (2.17) or (3.1), which gets rid of both the zero and the quasi-zero modes, and leaves the quantum fluctuations.

eq.(3.11) in fact gives a divergent field strength (and consequently a divergent Lagrangian density) near the origin, even though the action is a finite and well-behaved function of R. Satisfactory $I\bar{I}$ solutions with finite field strength everywhere are not hard to find. For example, we can choose[9]

$$A_{I\bar{I}} = \text{Im} \left\{ \frac{\frac{\bar{x}dx}{x^4} + \frac{\overline{(x-R)}dx}{(x-R)^2}}{1 + \frac{R^2}{x^2(1+R^2)} + \frac{1}{(x-R)^2}} \right\}. \tag{3.15}$$

We will continue to use eq.(3.11), however, not only because we consider the pathology a mild one, but also because it is easier to generalize it to $I^N\bar{I}^{\bar{N}}$ solutions. For those who are truly bothered by the divergence problem, eq.(3.11) and other formulas based on it in this papers could be viewed as a short-hand for better (but usually more complicated) solutions such as eq.(3.15).

4. Multiple Instantons and Antiinstantons

After dealing with $I\bar{I}$, the generalization to $I^N\bar{I}^{\bar{N}}$ is relatively straightforward. Again, we begin by setting up criteria. We find that they should read

1) $A_{I^N\bar{I}^{\bar{N}}}$ belongs in the $Q = N - \bar{N}$ sector.
2) $A_{I^N\bar{I}^{\bar{N}}}$ has easily identifiable instanton parameters, and covers the entire $8(N + \bar{N})$-dimensional parameter space spanned by all zero- and nonzero-modes.
3.1) If a subset $I^{N'}\bar{I}^{\bar{N}'}$ becomes widely separated from the rest, $A_{I^N\bar{I}^{\bar{N}}}$ reduces to the sum of $A_{I^{N'}\bar{I}^{\bar{N}'}}$ and $A_{I^{(N-N')}\bar{I}^{(\bar{N}-\bar{N}')}}$.
3.2) If subsets $I^{N'}$ and $\bar{I}^{N'}$ have identical sizes and positions, and are widely separated from the rest, they annihilate each other.
4) $A_{I^N\bar{I}^{\bar{N}}}$ respects the conformal symmetries.

This is rather straightforward once it is written down. The only thing that needs explanation is that we don't require $I^{N'}$ and $\bar{I}^{N'}$ to annihilate each other in the presence of other (anti)instantons. The reason is of course that in the non-trivial background field generated by other instantons, the parity between instantons and antiinstantons is broken. This is a manifestation of the nonlinear nature of the Yang-Mills theory.

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[9] We will ignore the phases a, b again. It is trivial to put the relative phase back at the end of our discussion if one chooses to.

Notice that because of $Cri.3.1$, $Cri.3.2$ is equivalent to

3.2′) If I^N and $\bar{I}^N$ have identical sizes and positions, $A_{I^N \bar{I}^N}$ approaches the trivial vacuum.

We will ignore the phases for now. Recall that the I^N solution with no phases can be written in the 't Hooft form[14–16],

$$A_{I^N}^{'\text{tHooft}} = \text{Im}\left\{\frac{\sum_{i=1}^{N} \frac{\rho_i^2 \overline{(x-R_i)}}{(x-R_i)^4}dx}{1+\sum_{i=1}^{N} \frac{\rho_i^2}{(x-R_i)^2}}\right\}. \tag{4.1}$$

This will be the analog of an instanton in the singular gauge. The analog of an antiinstanton in the regular gauge can be found by operating on an $\bar{I}^{\bar{N}}$ solution in the 't Hooft form the following gauge transformation,

$$g_0 = \frac{\sum_{i=1}^{\bar{N}} \frac{\rho_i'^2 (x-R_i')}{(x-R_i')^4}}{\left|\sum_{i=1}^{\bar{N}} \frac{\rho_i'^2 (x-R_i')}{(x-R_i')^4}\right|}, \tag{4.2}$$

where the "′" designates the parameters of the antiinstantons as compared to those of the instantons. We have

$$\begin{aligned} A_{\bar{I}^{\bar{N}}}^{g_0} &= g_0^{-1} A_{\bar{I}^{\bar{N}}}^{'\text{tHooft}} g_0 + g_0^{-1} dg_0 \\ &= \text{Im}\left\{ -\left(\frac{\sum_{i=1}^{\bar{N}} \frac{\rho_i'^2 \overline{(x-R_i')}}{(x-R_i')^4}dx}{1+\sum_{i=1}^{N} \frac{\rho_i'^2}{(x-R_i')^2}}\right) \right. \\ &\quad \left. +\frac{\left(\sum_{i=1}^{\bar{N}} \frac{\rho_i'^2 \overline{(x-R_i')}}{(x-R_i')^4}\right) d\left(\sum_{i=1}^{\bar{N}} \frac{\rho_i'^2 (x-R_i')}{(x-R_i')^4}\right)}{\left|\sum_{i=1}^{\bar{N}} \frac{\rho_i'^2 (x-R_i')}{(x-R_i')^4}\right|^2}\right\}. \end{aligned} \tag{4.3}$$

Now, clearly the first term in (4.3) exactly cancels (4.1) when the positions and sizes of the instantons are identical to those of the antiinstantons. Thus if we choose

$$A_{I^N \bar{I}^{\bar{N}}} = A_{I^N}^{'\text{tHooft}} + A_{\bar{I}^{\bar{N}}}^{g_{\bar{N}}}, \tag{4.4}$$

it will satisfy $Cri.3.2'$. In fact, it is easy to see that it also satisfies $Cri.3.1$ because if some (anti)instantons are far away, their contributions are suppressed by at least the inverse square of the distances, in both the numerator and the denominator of the expression.

As for the other criteria, $Cri.1$ and 2 are again satisfied trivially. $Cri.4$ requires more thought, though. Clearly (4.4) respects the translational and rotational symmetries. The special conformal transformations will introduce relative phases within any pair in either of the subsets I^N or $\bar{I}^{\bar{N}}$ unless the vector of the special conformal boost coincides with the axis of the I^2 ($\bar{I}^2$) pair[12]. Since we have assumed no relative phase so far, we don't have to worry about these special conformal transformations except for a few special cases, such as $I^2\bar{I}$ or when everything lines up in a straight line. In either case, one simply introduces any appropriate constraint to kill off the extra degree of freedom. The same can be easily done for dilitation also. Anyway, we can be excused for skimping the details concerning the dilitation and the special conformal symmetries because they are not present in the Yang-Mills-Higgs theory wherein our ultimate interest lies.

With (4.4), one may begin by computing $S(A_{I\bar{I}})$. Subtracting the "self-action" $2S_I$ from $S(A_{I\bar{I}})$ then gives the two-body interaction between an instanton-antiinstanton pair[10]. One then proceeds to compute $S(A_{I^2\bar{I}})$ and $S(A_{I\bar{I}^2})$. Subtracting the self-action and the two-body interactions between all pairs then gives the three-body interactions. This process can be carried over to yield the n-body interaction for any n. In practice, one may want to assume that these many body interactions become less and less important as n grows large.

We have given the expressions for the $I^N\bar{I}^{\bar{N}}$ valley configurations without phases. Now we will see how to introduce phases into them. The two overall phases a and b for I^N and $\bar{I}^{\bar{N}}$ respectively can be put into (4.4) in the same manner as in (3.3). The relative phases within I^N ($\bar{I}^{\bar{N}}$) are much harder to deal with, however. As readers familiar with ref.1 would know, the $8N-3$ physical parameters of the exact I^N solution are buried deep in a maze of quaternion matrix algebra. To interpret the positions, sizes and phases of even the simplest I^2 solution is not exactly a trivial task[12]. It is therefore not surprising to find that our linear construction of the $I^N\bar{I}^{\bar{N}}$ valley doesn't work with these solutions. More specifically, we are unable to find the suitable gauge transformation as in (4.2) which is vital for our solution to satisfy $Cri.3.2'$.

Although this looks very much like the end of the story, we in fact have another recourse to go to. This is the work of Jackiw, Nohl and Rebbi[16], in which they generalized

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[10] Note that because we have used the exact N-instanton solution in our construction, the interaction among any number of instantons remains zero. The same is true for antiinstantons.

the 't Hooft form to include more parameters, *i.e.*

$$A_{I^N}^{\text{JNR}} \;=\; \text{Im} \left\{ \frac{\sum_{i=0}^{N} \frac{\lambda_i^2 \overline{(x-r_i)}}{(x-r_i)^4} dx}{\sum_{i=0}^{N} \frac{\lambda_i^2}{(x-r_i)^2}} \right\}. \tag{4.5}$$

We shall call this the JNR gauge because it is gauge-equivalent to other forms of the I^N solution. Notice that the overall scale of λ's gets canceled between the numerator and the denominator, so there seems to be a total of $5N + 4$ parameters now. More careful examination reveals that some of these parameters correspond to gauge degrees of freedom for $N \leq 2$, so the actual numbers of independent parameters are 5 and 13 for $N = 1$ and 2 respectively.

Although it is not obvious from looking at (4.5), the extra parameters it carries compared to the 't Hooft form in fact correspond to relative phases[12]. Amazingly, (4.5) doesn't contain any quaternion matrices, and the analog of g_0 as in (4.2) can indeed be found. A discussion similar to what we did with the 't Hooft form then follows. We again skimp the details for the following reasons. The algebra is very messy and not inspiring at all. The problem it solves is not particularly important either, since when we evaluate a path integral, the integral over the phases can usually be approximated with the group volume. Besides, for large N's, (4.5) clearly doesn't have enough parameters to cover all the phases. We therefore simply state without proving the following result. Satisfactory expressions for $I^2 \bar{I}^2$ and $I^3 \bar{I}^3$ covering the entire parameter space can be found using (4.5). It may seem strange at first that it would work for $I^3 \bar{I}^3$, since the JNR form (4.5) is 2 parameters short for the entire space of I^3. Fortunately the conformal degrees of freedom are more than enough to make up for the difference.

Appendix A. Quaternions

Similar to its $\mathcal{C}$-number cousin $z = z_0 + iz_1$, a quaternion $x \in \mathcal{H}$ and its conjugate $\bar{x}$ are given by

$$x = x_0 + ix_1 + jx_2 + kx_3, \tag{A.1a}$$

$$\bar{x} = x_0 - ix_1 - jx_2 - kx_3, \tag{A.1b}$$

where $x_\mu \in \mathcal{R}$, and $\{i, j, k\}$ satisfy

$$i^2 = j^2 = k^2 = -1,$$
$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j. \tag{A.2}$$

Clearly the quaternion algebra has a 2×2 complex matrix representation :

$$\{1, i, j, k\} \ \rightarrow \ \{I, i\vec{\sigma}\}, \tag{A.3}$$

where σ_m are the Pauli matrices. Therefore the group $SU(2)$ can be identified with $SP(1)$, the group of unit quaternions, and the $SU(2)$ algebra correspond to Im $\mathcal{H}$.

One can also identify $\mathcal{R}^4$ with $\mathcal{H}$ via (A.1a), and the $SU(2)$ gauge field $A_\mu(x)$ is then obviously a function of quaternions with imaginary quaternion values. When working with Yang-Mills instantons, we find that the notation can be even further simplified if we consider the one-form

$$A(x) \ = \ \sum_{\mu=0}^{3} A_\mu(x) dx^\mu. \tag{A.4}$$

The BPST instanton traditionally expressed in terms of the 't Hooft η tensor as

$$A_\mu(x) \ = \ \sum_{\mu=0}^{3} \frac{\sigma^m \eta_{m\mu\nu} x^\nu}{i(x^2 + \rho^2)}, \tag{A.5}$$

can now be written as

$$A_I(x) \ = \ \text{Im} \left\{ \frac{x d\bar{x}}{x^2 + \rho^2} \right\}, \tag{A.6}$$

and the antiinstanton is

$$A_{\bar{I}}(x) \ = \ \text{Im} \left\{ \frac{\bar{x} dx}{x^2 + \rho^2} \right\}. \tag{A.7}$$

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It is possible to do computations in the quaternion notation. For example, one may wish to evaluate the curvature 2–form F for the gauge field defined in (A.6). It is given by

$$
\begin{aligned}
F &= dA + A \wedge A \\
&= \operatorname{Im}\left\{\left(\frac{dx \wedge d\bar{x}}{x^2 + \rho^2} - \frac{xd(x^2 + \rho^2) \wedge d\bar{x}}{(x^2 + \rho^2)^2}\right) + \frac{xd\bar{x} \wedge xd\bar{x}}{(x^2 + \rho^2)^2}\right\} \\
&= \operatorname{Im}\left\{\frac{dx \wedge d\bar{x}}{x^2 + \rho^2} - \frac{x(d\bar{x}x + \bar{x}dx) \wedge d\bar{x}}{(x^2 + \rho^2)^2} + \frac{xd\bar{x} \wedge xd\bar{x}}{(x^2 + \rho^2)^2}\right\} \\
&= \frac{\rho^2 dx \wedge d\bar{x}}{(x^2 + \rho^2)^2}.
\end{aligned}
\tag{A.8}
$$

We dropped the Im symbol in the final expression because it is already pure imaginary.

A slightly more complicated example is to examine how (A.6) transforms under a special conformal boost, which can be defined as

$$
x \;\to\; x' \;=\; (x + a)(1 - \bar{a}x)^{-1}.
\tag{A.9}
$$

We begin by inversing (A.9),

$$
x \;=\; (1 + x'\bar{a})^{-1}(x' - a) \;=\; (x' - a)(1 + \bar{a}x')^{-1}.
\tag{A.10}
$$

Substituting (A.10) into (A.6), one finds that

$$
A_I(x) \;=\; \operatorname{Im}\left\{\frac{(1 + x'\bar{a})^{-1}(x' - a)d\left[(\bar{x}' - \bar{a})(1 + x'\bar{a})\right]}{(x' - a)^2 + \rho^2(1 + x'\bar{a})^2}\right\}.
\tag{A.11}
$$

This can be simplified with a gauge transformation,

$$
g \;=\; \frac{1 + a\bar{x}'}{\left|1 + a\bar{x}'\right|}.
\tag{A.12}
$$

We have

$$
\begin{aligned}
A \to A' &= g^{-1}Ag + g^{-1}dg \\
&= \operatorname{Im}\left\{\frac{(x' - a)d\left[(\bar{x}' - \bar{a})(1 + x'\bar{a})\right](1 + x'\bar{a})^{-1}}{(x' - a)^2 + \rho^2(1 + x'\bar{a})^2} + \frac{(1 + x'\bar{a})ad\bar{x}'}{(1 + a\bar{x}')^2}\right\} \\
&= \operatorname{Im}\left\{\frac{(x' - a)d\bar{x}'}{(x' - a)^2 + \rho^2(1 + x'\bar{a})^2} - \frac{(x' - a)^2(1 + x'\bar{a})ad\bar{x}'}{(1 + a\bar{x}')^2[(x' - a)^2 + \rho^2(1 + x'\bar{a})^2]}\right. \\
&\qquad\left. + \frac{(1 + x'\bar{a})ad\bar{x}'}{(1 + a\bar{x}')^2}\right\} \\
&= \operatorname{Im}\left\{\frac{(x' - a)d\bar{x}'}{(x' - a)^2 + \rho^2(1 + x'\bar{a})^2} + \frac{\rho^2(1 + x'\bar{a})ad\bar{x}'}{(x' - a)^2 + \rho^2(1 + x'\bar{a})^2}\right\} \\
&= \operatorname{Im}\left\{\frac{(x' - R)d\bar{x}'}{(x' - R)^2 + \rho'^2}\right\},
\end{aligned}
\tag{A.13}
$$

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where

$$R = \frac{(1 - \rho^2)a}{1 + \rho^2 a^2} \quad \text{and} \quad \rho' = \frac{(1 + a^2)\rho}{1 + \rho^2 a^2}. \tag{A.14}$$

This gives how the parameters of a single instanton change under the special conformal transformation.

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